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Discrete Symmetry Breaking for Certain Short-Range Interactions

M. Fukuyama[1]

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Georgii's theorem ensures that, restricted to two-dimensional planes, a single ocean (i.e., infinite connected component) of a ground state and islands (i.e., finite connected components) are observed in lattice spin systems at sufficiently low temperature. This paper extends his results for higher dimensional hyperplanes. Our proof is mainly based on a kind of Peierls argument and is different from Georgii's, which relies on the percolation method.

KEY WORDS: Gibbs measure; phase transition; lattice spin system; discrete symmetry breaking.

1. INTRODUCTION

A theorem due to Georgii (Theorem (18.25) in ref. 1) concerns phase transitions in classical lattice spin systems with short-range interaction. When the dimension of the underlying lattice $\mathbb{Z}^d$ is two, from his theorem, one can construct Gibbs measures with the following feature at sufficiently low temperature. With respect to each Gibbs measure, with probability one, there is a unique infinite connected region of the corresponding ground state whose complement is composed of only finite connected components. One can expect that the same result holds for the lattice $\mathbb{Z}^d$ ($d \geqslant 3$) and, indeed, we shall prove it in the main theorem (Theorem 1).

To explain the difference between the original theorem due to Georgii and ours, let us apply these two results to the simple case, namely the

[1] Graduate School of Mathematical Sciences, University of Tokyo, Tokyo 153-8914, Japan; e-mail: fuquyama@interlink.or.jp.

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well-known Ising model. The Ising model on the lattice $\mathbb{Z}^d$ is described by the Hamiltonian

$$H(\omega) = -\sum_{\substack{\{i,\, j\} \subset \mathbb{Z}^d \\ \|i-j\| = 1}} \omega_i \omega_j, \qquad \omega = (\omega_j)_{j \in \mathbb{Z}^d} \in \{\pm 1\}^{\mathbb{Z}^d}$$

For the two-dimensional Ising model, two theorems give the same result: If the temperature is sufficiently low, there exist Gibbs measures μ_+ and μ_- which satisfy the following property. With respect to the Gibbs measure μ_+ (respectively μ_-), with probability one, there is a unique infinite connected region of $+1$ spins (respectively -1 spins) whose complement is composed of only finite connected components. In other words, we observe "an ocean and islands" of $+1$ spins (respectively -1 spins) under μ_+ (respectively μ_-) on the whole lattice $\mathbb{Z}^2$.

On the other hand, in the case of three or more than three-dimensional Ising model, they give different results. The original theorem constructs Gibbs measures μ_+^P and μ_-^P depending on each given two-dimensional plane P of $\mathbb{Z}^d$. With respect to μ_+^P and μ_-^P, "the ocean and islands" can be observed only on the plane P, not on the whole lattice $\mathbb{Z}^d$. It is however natural to expect the existence of Gibbs measures for which "the ocean and islands" are observed on the whole lattice $\mathbb{Z}^d$.

The aim of this paper is to construct such Gibbs measures in higher dimensions. Georgii used certain topological property of two-dimensional plane $\mathbb{Z}^2$ such as the plane can be separated into two parts by a connected path with infinite length. We shall use different property of hyperplane $\mathbb{Z}^d$ ($d \geqslant 2$), see Corollary 1 and Corollary 2 below.

Before going on to the main subject, we shall summarize in the next section some notations and terminology mainly according to Chapter 17 and Chapter 18 in ref. 1.

2. NOTATIONS AND MODELS

We consider lattice spin systems on $\mathbb{Z}^d$ ($d \geqslant 2$) where each spin variable takes values in a measure space $(E, \mathscr{E}, \lambda)$. We assume that $(E, \mathscr{E})$ is a standard Borel space and $\lambda(E) < \infty$. Let $(\Omega, \mathscr{F})$ denote the configuration space $(E, \mathscr{E})^{\mathbb{Z}^d}$. We denote by σ_Λ the restriction for each spin configuration $\omega \in \Omega$ to the components belonging to $\Lambda \subset \mathbb{Z}^d$, i.e., $\sigma_\Lambda(\omega) = (\omega_i)_{i \in \Lambda}$. For each $\Delta \subset \mathbb{Z}^d$, we define the σ-algebra $\mathscr{F}_\Delta$ by

$$\mathscr{F}_\Delta \triangleq \sigma[\sigma_\Lambda; \Lambda \subset \Delta, \sharp\Lambda < \infty]$$

Let $C = \{0, 1\}^d$ be a unit cube in $\mathbb{Z}^d$ and $\mathscr{L} = \{\Lambda \subset \mathbb{Z}^d \mid 0 < \sharp\Lambda < \infty\}$.

Hamiltonians. We call a bounded measurable function $\Phi: E^C \to \mathbb{R}$ an interaction potential. For an interaction potential Φ and $\Lambda \in \mathscr{L}$ we define the Hamiltonian $H_\Lambda^\Phi: \Omega \to \mathbb{R}$ by

$$H_\Lambda^\Phi(\omega) \triangleq \sum_{i:\, C(i) \cap \Lambda \neq \varnothing} \Phi(\sigma_{C(i)}(\omega)) \tag{1}$$

Here $C(i) = C + i$ is the translation of C by $i \in \mathbb{Z}^d$. In the expression $\Phi(\sigma_{C(i)}(\omega))$, we identify $E^{C(i)}$ with E^C in the natural way. We remark that, by definition, the interactions of the spin systems are short-range and translation invariant.

Gibbs Measures. The finite volume Gibbs measure for Φ in $\Lambda \in \mathscr{L}$ with boundary condition $\omega \in \Omega$ is the probability measure $\gamma_\Lambda^\Phi(\cdot \mid \omega)$ on $(\Omega, \mathscr{F})$ defined by

$$\begin{aligned} \gamma_\Lambda^\Phi(A \mid \omega) \triangleq (Z_\Lambda^\Phi(\omega))^{-1} \int \lambda^\Lambda(d\zeta) \exp[-H_\Lambda^\Phi(\zeta\omega_{\mathbb{Z}^d \setminus \Lambda})] \\ \times 1_A(\zeta\omega_{\mathbb{Z}^d \setminus \Lambda}) \qquad (A \in \mathscr{F}) \end{aligned} \tag{2}$$

where $Z_\Lambda^\Phi(\omega)$ is the normalization factor, $\lambda^\Lambda(d\zeta) = \prod_{i \in \Lambda} \lambda(d\zeta_i)$ and $\zeta\omega_{\mathbb{Z}^d \setminus \Lambda} \in \Omega$ is defined by

$$(\zeta\omega_{\mathbb{Z}^d \setminus \Lambda})_i = \begin{cases} \zeta_i & \text{if} \quad i \in \Lambda \\ \omega_i & \text{if} \quad i \in \mathbb{Z}^d \setminus \Lambda \end{cases}$$

A Gibbs measure μ for Φ is defined as a probability measure μ on $(\Omega, \mathscr{F})$ which satisfies

$$\mu(A \mid \mathscr{F}_{\mathbb{Z}^d \setminus \Lambda}) = \gamma_\Lambda^\Phi(A \mid \cdot) \quad \mu\text{-a.s.} \qquad \text{for all} \quad A \in \mathscr{F} \text{ and } \Lambda \in \mathscr{L} \tag{3}$$

where the left-hand side stands for a conditional probability. The set of all Gibbs measures for Φ is denoted by $\mathscr{G}(\Phi)$.

To investigate the dependence on temperature, throughout this paper, we study the set $\mathscr{G}(\beta\Phi)$ with the parameter $\beta > 0$ which is inverse temperature.

Symmetries. In order to describe symmetries of the interactions Φ we introduce two classes of transformations of $(E, \mathscr{E})^C$; reflections and pure spin transformations.

(i) Reflections $(r_k)_{1 \leqslant k \leqslant d}$: r_k is the spatial reflection of $\omega = (\omega_i)_{i \in C} \in E^C$ with respect to the hyperplane $\{i \in \mathbb{R}^d \mid i_k = \frac{1}{2}\}$ defined by

$$(r_k \omega)_j \triangleq \omega_{R_k j} \qquad \text{and} \qquad (R_k j)_l \triangleq \begin{cases} 1 - j_l & \text{if} \quad l = k \\ j_l & \text{otherwise} \end{cases} \qquad (1 \leqslant l \leqslant d) \tag{4}$$

for each $j \in C$.

(ii) Pure spin transformations: A transformation τ of $(E, \mathscr{E})^C$ is called a pure spin transformation if $\tau\omega$ $(\omega \in E^C)$ can be written as

$$\tau\omega = (\tau_i \omega_i)_{i \in C} \tag{5}$$

where τ_i's are invertible transformations of $(E, \mathscr{E})$ such that $\lambda \circ \tau_i^{-1} = \lambda$.

Each transformation defined in (i) and (ii) can be extended in a natural way to that of $(\Omega, \mathscr{F})$, which are denoted by the same letter r_k and τ, respectively. In fact, the reflection transformation r_k of $(\Omega, \mathscr{F})$ is defined by the same form (4), while the pure spin transformation $\tau = (\tau_i)_{i \in \mathbb{Z}^d}$ of $(\Omega, \mathscr{F})$ is defined by

$$\tau_i = \tau_j \qquad \text{if} \quad i_k \equiv j_k \bmod 2 \qquad \text{for all} \quad 1 \leqslant k \leqslant d$$

Let T denote the transformation group on $(E, \mathscr{E})^C$ or $(\Omega, \mathscr{F})$ which is generated by the transformations (i) and (ii). For each interaction potential Φ, let $I(\Phi)$ be the subgroup of T which consists of all transformations $\tau \in T$ under which Φ is invariant, i.e., $\Phi \circ \tau^{-1} = \Phi$.

Domains of $G \in \mathscr{E}^C$ in Each Spin Configuration ω. We consider a $(d-p)$-dimensional hyperplane P of the form

$$P = \{i \in \mathbb{Z}^d \mid i_{k_1} = a_1, i_{k_2} = a_2, \ldots, i_{k_p} = a_p\} \qquad (0 \leqslant p \leqslant d-2)$$

where $a_1, a_2, \ldots, a_p \in \mathbb{Z}$ and $1 \leqslant k_1 < k_2 < \cdots < k_p \leqslant d$. Note that if $p = 0$ then P denotes the whole lattice $\mathbb{Z}^d$. Since the Hamiltonians (1) are translation invariant, we can assume that the hyperplane P contains the origin O without loss of generality. That is,

$$P = \{i \in \mathbb{Z}^d \mid i_{k_1} = 0, i_{k_2} = 0, \ldots, i_{k_p} = 0\} \qquad (0 \leqslant p \leqslant d-2) \tag{6}$$

For each $G \in \mathscr{E}^C$ and $\omega \in \Omega$, the domains of G for ω in P, which is denoted by $V_P(G, \omega)$, is defined by

$$V_P(G, \omega) \triangleq \{i \in P \mid \sigma_C(\theta_{-i}\omega) \in r^i G\} \tag{7}$$

where $\theta_i\omega \in \Omega$ is the translation of $\omega \in \Omega$ defined by $(\theta_i\omega)_j = \omega_{j-i}$ for each $j \in \mathbb{Z}^d$ and $r^i = r_1^{i_1} \circ r_2^{i_2} \circ \cdots \circ r_d^{i_d}$ $(i = (i_1,..., i_d) \in \mathbb{Z}^d)$ with reflections $(r_k)_{1 \leqslant k \leqslant d}$ introduced above.

The Ocean of *G* in Each Spin Configuration ω. We need two norms $|\cdot|$ and $\|\cdot\|$ on $\mathbb{Z}^d$ respectively defined by

$$|i| = \max_{1 \leqslant k \leqslant d} |i_k| \qquad \text{and} \qquad \|i\| = \sum_{k=1}^{d} |i_k| \qquad (i \in \mathbb{Z}^d)$$

For each subset A of $\mathbb{Z}^d$ we say that A is connected if for any $i, j \in A$ there exists a finite sequence $\{i^{(0)},..., i^{(n)}\}$ in A such that $i^{(0)} = i$, $\|i^{(m)} - i^{(m+1)}\| = 1$ $(0 \leqslant m \leqslant n-1)$ and $i^{(n)} = j$. Replacing $\|\cdot\|$ with $|\cdot|$, we define the term "$*$connected" similarly.

For each $G \in \mathscr{E}^C$ and $\omega \in \Omega$, we let $\xi_P(G, \omega)$ denote the union of all infinite connected components of $V_P(G, \omega)$. Further we define $\xi_P^0(G, \omega)$ by

$$\xi_P^0(G, \omega) \triangleq \begin{cases} \xi_P(G, \omega) & \text{if } \xi_P(G, \omega) \text{ is connected and all } *\text{connected} \\ & \text{components of } P \backslash \xi_P(G, \omega) \text{ are finite sets} \\ \varnothing & \text{otherwise} \end{cases} \tag{8}$$

We should notice that $\xi_P^0(G, \omega)$ stands for "the ocean of G and islands" on the hyperplane P. The following lemma concerns a property of $\xi_P^0(G, \omega)$. It follows by standard arguments from Lemma 4 and Corollary 2 stated below.

Lemma 1. For each $G \in \mathscr{E}^C$, the event $\{\xi_P^0(G, \cdot) \neq \varnothing\}$ is measurable with respect to the tail σ-algebra $\mathscr{T} \triangleq \bigcap_{\Lambda \in \mathscr{L}} \mathscr{F}_{\mathbb{Z}^d \backslash \Lambda}$.

Local Ground States. For each $\varepsilon \geqslant 0$ and interaction potential Φ, the set $G_\varepsilon(\Phi) \in \mathscr{E}^C$ is defined by

$$G_\varepsilon(\Phi) \triangleq \{\omega \in E^C \mid \Phi(\omega) \leqslant m_\Phi + \varepsilon\} \tag{9}$$

where $m_\Phi \triangleq \sup\{t \in \mathbb{R} \mid \lambda^C(\Phi \leqslant t) = 0\}$. Roughly speaking, $G_\varepsilon(\Phi)$ denotes the set of all local ground states.

Now, we shall proceed to our main theorem.

3. MAIN THEOREM

For each hyperplane P of the form (6), we introduce a condition $[P]$ for interaction potentials Φ.

Condition [P]:

(i) Φ has N distinct sets $G_1, G_2,..., G_N \in \mathscr{E}^C$ satisfying the following three conditions.

(a) For any sufficiently small $\varepsilon > 0$,

$$G_\varepsilon(\Phi) \subset G_1 \cup G_2 \cup \cdots \cup G_N$$

(b) For each subset $F = \{i \in C \mid i_k = a\}$ of C with $a \in \{0, 1\}$ and kth axis contained in P (they are called P-faces in ref. 1),

$$\text{the sets } (G_1)_F, (G_2)_F,..., (G_N)_F \text{ are pairwise disjoint}$$

Here $(G)_F = \{(\omega_i)_{i \in F} \in E^F \mid \omega \in G\}$ for $G \in \mathscr{E}^C$, which is the restriction of G to F.

(c) For each $1 \leqslant m, n \leqslant N$, there exists some $\tau \in I(\Phi)$ with $\tau G_m = G_n$.

(ii) $r_k \in I(\Phi)$ for all $1 \leqslant k \leqslant d$.

The following is our main theorem.

Theorem 1. Let Φ be any interaction potential satisfying the condition [P] for any given hyperplane P of the form (6).

(i) For any sufficiently large β, there exist N distinct Gibbs measures $\mu_1, \mu_2,..., \mu_N \in \mathscr{G}(\beta\Phi)$ such that

$$\mu_n(\xi^0_P(G_n, \cdot) \neq \varnothing) = 1 \qquad (1 \leqslant n \leqslant N)$$

(ii) These measures $(\mu_n)_{1 \leqslant n \leqslant N}$ enjoy, in addition, the following properties.

(a) $\lim_{\beta \to \infty} \mu_n(\xi^0_P(G_n, \cdot) \ni O) = 1$.

(b) For all $\tau \in I(\Phi)$ satisfying $\tau G_m = G_n$, $\tau\mu_m = \mu_n$ hold. In particular, if $r_k G_m = G_n$ for some reflection $r_k \in I(\Phi)$ with kth axis contained in P, then in addition $\theta_{u_k}\mu_m = \mu_n$ hold, where u_k is the unit vector in direction k. Here $\tau\mu_m$ and $\theta_{u_k}\mu_m$ are defined by $\tau\mu_m = \mu_m \circ \tau^{-1}$ and $\theta_{u_k}\mu_m = \mu_m \circ \theta_{-u_k}$, respectively.

We notice that the Gibbs measures $(\mu_n)_{1 \leqslant n \leqslant N}$ depend on the hyperplane P. We suppress P-dependence of the measures.

Remark. (1) The statement (ii)(b) contains the case $m = n$, which gives symmetries of $(\mu_n)_{1 \leqslant n \leqslant N}$.

(2) The Georgii's original theorem (Theorem (18.25) in ref. 1) is a special case of Theorem 1 that P is a two-dimensional plane. When the dimension of P is more than two, the assertion of Theorem 1 does not follow from the original theorem. Theorem 1 is an extension of the original theorem for higher dimensional hyperplanes P.

(3) We emphasize Theorem 1 holds in particular for $P = \mathbb{Z}^d$. The case where the dimension of P is less than d is also interesting. Such an example is given in Section 18.3.10. of ref. 1, where P is a two-dimensional plane in the lattice $\mathbb{Z}^3$.

The proof of Theorem 1 will be given in the next section.

Example. Now we apply Theorem 1 to the Ising model whose lattice dimension is at least two. The space of spin variables is $E = \{\pm 1\}$ and the Hamiltonian (1) is given by the interaction potential Φ: $E^C \to \mathbb{R}$ defined as

$$\Phi(\omega) = -2^{-d} \sum_{\substack{\{i, j\} \subset C \\ \|i-j\| = 1}} \omega_i \omega_j \qquad \text{for} \quad \omega \in E^C \tag{10}$$

We introduce the spin-flip transformation τ_{rev} of $(E, \mathscr{E})^C$ by

$$\tau_{\mathrm{rev}} \omega = (-\omega_i)_{i \in C} \qquad \text{for} \quad \omega \in E^C$$

Let $G_\pm = \{\omega_\pm\} \in \mathscr{E}^C$, where $(\omega_\pm)_i = \pm 1$ $(i \in C)$. We have $G_\pm = \tau_{\mathrm{rev}} G_\mp$. It can be easily checked that the potential Φ satisfies the condition $[P]$ with G_+, G_- and $\tau_{\mathrm{rev}} \in I(\Phi)$ for any hyperplane P of the form (6).

Thus, by Theorem 1, we have the following result for any fixed hyperplane P of the form (6). For any sufficiently large β, there exist two Gibbs measures $\mu_+, \mu_- \in \mathscr{G}(\beta\Phi)$ such that

$$\mu_\pm(\xi_P^0(G_\pm, \cdot) \neq \emptyset) = 1$$

Furthermore the measures μ_+ and μ_- satisfy

$$\lim_{\beta \to \infty} \mu_\pm(\xi_P^0(G_\pm, \cdot) \ni O) = 1$$

and

$$\tau_{\mathrm{rev}} \mu_\pm = \mu_\mp$$

We remark that this result is not a new result. The Pirogov–Sinai theory implies this result immediately for $P = \mathbb{Z}^d$ and with certain additional small arguments for general hyperplanes P, see ref. 3.

4. PROOF OF THEOREM 1

In this section we shall give the proof of Theorem 1. In the first half, we shall mainly follow the argument in the proof of Theorem (18.25) in ref. 1. We fix a hyperplane P of the form (6). We let Φ be an interaction potential which satisfies the condition $[P]$.

We start with constructing the Gibbs measures $(\mu_n)_{1 \leqslant n \leqslant N}$ in the same way as Theorem (18.25) in ref. 1. Let ${}^{\circ}\gamma^{\Phi}_{\Lambda_l}$ denote the finite volume Gibbs measure in $\Lambda_l \triangleq [-l, l]^d \cap \mathbb{Z}^d$ for Φ with periodic boundary condition defined by

$$ {}^{\circ}\gamma^{\Phi}_{\Lambda_l}(A) \triangleq ({}^{\circ}Z^{\Phi}_{\Lambda_l})^{-1} \int_A \lambda^{\Lambda_l}(d\zeta) \exp\left[-\sum_{i \in \Lambda_l} \Phi(\sigma_{C(i)}(\tilde{\zeta}))\right] \qquad (A \in \mathscr{E}^{\Lambda_l}) $$

Here ${}^{\circ}Z^{\Phi}_{\Lambda_l}$ is the normalization factor and $\tilde{\zeta} \in \Omega$ is the periodic continuation of $\zeta \in E^{\Lambda_l}$, i.e., $(\tilde{\zeta})_i \triangleq \zeta_{j(i)}$ $(i \in \mathbb{Z}^d)$, where $j(i)$ is the unique element of Λ_l with $j(i) = i \bmod \Lambda_1$.

We denote by $\mathscr{G}_0(\beta\Phi)$ the set of all cluster points in the $\mathscr{L}$-topology (see (4.2) in ref. 1) of any sequence of probability measures $({}^{\circ}\gamma^{\beta\Phi}_{\Lambda_l} \times \delta_{\omega_l})_{l \geqslant 1}$ of $(\Omega, \mathscr{F})$, where δ_{ω_l} is the Dirac measure on $(E, \mathscr{E})^{\mathbb{Z}^d \setminus \Lambda_l}$ at an arbitrary $\omega_l \in E^{\mathbb{Z}^d \setminus \Lambda_l}$.

$\mathscr{G}_0(\beta\Phi)$ has the following three properties.

- $\mathscr{G}_0(\beta\Phi) \neq \varnothing$.
- $\mathscr{G}_0(\beta\Phi) \subset \mathscr{G}(\beta\Phi)$.
- Each $\mu \in \mathscr{G}_0(\beta\Phi)$ is invariant under $\tau \in I(\Phi)$.

The first property follows from the fact that $(E, \mathscr{E})$ is a standard Borel space, see Proposition (18.12) in ref. 1.

Now we take an arbitrary Gibbs measure μ from $\mathscr{G}_0(\beta\Phi)$ and construct probability measures $(\mu_n)_{1 \leqslant n \leqslant N}$ as

$$ \mu_n \triangleq \mu(\cdot \mid A_n) \qquad \text{and} \qquad A_n \triangleq \{\xi^0_P(G_n, \cdot) \neq \varnothing\} \qquad (1 \leqslant n \leqslant N) \tag{11} $$

where $\mu(\cdot \mid A_n)$ stands for the elementary conditional probability $\mu(\cdot \cap A_n)/\mu(A_n)$.

By Lemma 1, each A_n is a tail event, which implies that $\mu_n \in \mathscr{G}(\beta\Phi)$. We assert that these Gibbs measures $(\mu_n)_{1 \leqslant n \leqslant N}$ enjoy all the properties of Theorem 1. In fact, its proof goes quite similarly to that of Theorem (18.25) in ref. 1, and what we really need to show, except for the property (ii)(a), is that each μ_n is well-defined, that is the positivity of $\mu(A_n)$. Note that the condition $[P]$ ensures the following three properties (the proof of Theorem (18.25) in ref. 1 essentially works also in our setting).

- $\{\xi_P^0(G_\varepsilon(\Phi), \cdot) \neq \emptyset\} \subset \bigcup_{n=1}^N A_n$.
- $A_n \cap A_m = \emptyset$ for all distinct n, m.
- $\mu(A_1) = \mu(A_2) = \cdots = \mu(A_N)$ for all $\mu \in \mathscr{G}_0(\beta\Phi)$.

So the following proposition, which implies the positivity of $\mu(\xi_P^0(G_\varepsilon(\Phi), \cdot) \neq \emptyset)$, ensures the positivity of $\mu(A_n)$. This proposition also completes the proof of the property (ii)(a) in Theorem 1 in the same way as Theorem (18.17) completes that of Theorem (18.25) in ref. 1.

Proposition 1. For each $\varepsilon > 0$ and $\eta > 0$, there exists some $\beta_0 < \infty$ such that

$$\mu(\xi_P^0(G_\varepsilon(\Phi), \cdot) \ni O) \geqslant 1 - \eta$$

for each $\beta > \beta_0$ and $\mu \in \mathscr{G}_0(\beta\Phi)$.

To prove Proposition 1, we put $\Lambda_L^P = \{i \in P \mid |i| \leqslant L\}$ and

$$A_L^P \triangleq \{\omega \in \Omega \mid P \backslash V_P(G_\varepsilon(\Phi), \omega) \text{ has no } *\text{connected components } \Gamma \text{ satisfying } \Lambda_L^P \cap \Gamma \neq \emptyset \text{ and } \sharp\Gamma > \log L\}$$

Proposition 1 immediately follows from the following two propositions.

Proposition 2. There exists a function $z(\varepsilon, \beta)$ of $\varepsilon > 0$ and $\beta > 0$, which satisfies the following two conditions.

(i) $\mu(\bigcap_{L=1}^\infty A_L^P) \geqslant 1 - z(\varepsilon, \beta)$ for each $\mu \in \mathscr{G}_0(\beta\Phi)$.

(ii) $\lim_{\beta\to\infty} z(\varepsilon, \beta) = 0$ for each $\varepsilon > 0$.

Proposition 3.

$$\bigcap_{L=1}^\infty A_L^P \subset \{\xi_P^0(G_\varepsilon(\Phi), \cdot) \ni O\} \qquad \text{for each} \quad \varepsilon > 0$$

To prove Proposition 2, we need the following two lemmas.

Lemma 2 (Georgii[1,2]). There is a positive number K such that for each finite subset D of the hyperplane P and $\mu \in \mathscr{G}_0(\beta\Phi)$

$$\mu(D \cap V_P(G_\varepsilon(\Phi), \cdot) = \emptyset) \leqslant t(\varepsilon, \beta)^{\sharp D} \tag{12}$$

where $t(\varepsilon, \beta) = Ke^{-\varepsilon\beta/2}$.

This is taken from Lemma (18.10) in ref. 1 in the form adapted to our situation. The condition $[P]$(ii) is essential for the proof.

Lemma 3 (Sinai[3]). Let j be a fixed site of $\mathbb{Z}^d$. The number of $*$connected subsets Γ of $\mathbb{Z}^d$ such that $j \in \Gamma$ and $\sharp\Gamma = \gamma$ is at most $e^{M\gamma}$, where M is a suitable positive constant.

This is a special case of Lemma 2.7 in ref. 3.

Proof of Proposition 2. To simplify description, without loss of generality we assume that P is the whole lattice $\mathbb{Z}^d$ $(d \geqslant 2)$. The general case follows by the same argument. For notational convenience we use simple notations $V(G, \omega)$, Δ_L and A_L instead of $V_{\mathbb{Z}^d}(G, \omega)$, $\Delta_L^{\mathbb{Z}^d}$ and $A_L^{\mathbb{Z}^d}$, respectively, which means $\Delta_L = [-L, L]^d \cap \mathbb{Z}^d$ and

$$A_L = \{\omega \in \Omega \mid \mathbb{Z}^d \backslash V(G_\varepsilon(\Phi), \omega) \text{ has no } *\text{connected components } \Gamma \text{ satisfying} \\ \Delta_L \cap \Gamma \neq \varnothing \text{ and } \sharp\Gamma > \log L\}$$

Also we assume that Φ satisfies the condition $[\mathbb{Z}^d]$.

We notice that the following two statements are equivalent:

(i) $Z^d \backslash V(G_\varepsilon(\Phi), \omega)$ has a $*$connected component Γ such that $\Delta_L \cap \Gamma \neq \varnothing$ and $\sharp\Gamma > \log L$.

(ii) There exists some $*$connected subset D of $\mathbb{Z}^d$ such that

$$D \cap V(G_\varepsilon(\Phi), \omega) = \varnothing,\ \Delta_L \cap D \neq \varnothing \text{ and } \sharp D = [\log L] + 1$$

We let $\mathscr{D}_L$ denote the family of all $*$connected subsets D of $\mathbb{Z}^d$ satisfying $\Delta_L \cap D \neq \varnothing$ and $\sharp D = [\log L] + 1$. Then we can write

$$\Omega \backslash A_L = \bigcup_{D \in \mathscr{D}_L} \{D \cap V(G_\varepsilon(\Phi), \cdot) = \varnothing\}$$

Therefore

$$\begin{aligned}
\mu\left(\bigcap_{L=1}^{\infty} A_L\right) &\geqslant 1 - \sum_{L=1}^{\infty} \mu(\Omega \backslash A_L) \\
&\geqslant 1 - \sum_{L=1}^{\infty} \sum_{D \in \mathscr{D}_L} \mu(D \cap V(G_\varepsilon(\Phi), \cdot) = \varnothing) \\
&\geqslant 1 - \sum_{L=1}^{\infty} (2L+1)^d\, e^{M([\log L]+1)} t(\varepsilon, \beta)^{[\log L]+1} \\
&= 1 - \sum_{L=1}^{\infty} (2L+1)^d\, (K e^{M - \varepsilon\beta/2})^{[\log L]+1}
\end{aligned}$$

Here in the third inequality we use Lemma 2 and the estimate

$$\sharp\mathscr{D}_L \leqslant (2L+1)^d \, e^{M([\log L]+1)}$$

which follows from Lemma 3.

By the way if we pick a constant β_0 so large that $M+\log K+d-\varepsilon\beta_0/2<-1$, for any $\beta>\beta_0$ we have

$$\begin{aligned}
&\sum_{L=1}^{\infty} (2L+1)^d \, (Ke^{M-\varepsilon\beta/2})^{[\log L]+1} \\
&\qquad \leqslant \sum_{L=1}^{\infty} 3^d e^{d\log L+(M+\log K-\varepsilon\beta/2)([\log L]+1)} \\
&\qquad \leqslant \sum_{L=1}^{\infty} 3^d e^{(M+\log K+d-\varepsilon\beta_0/2)\log L} \\
&\qquad = \sum_{L=1}^{\infty} 3^d L^{M+\log K+d-\varepsilon\beta_0/2} < \infty
\end{aligned} \tag{13}$$

Therefore if we set

$$z(\varepsilon, \beta) = \sum_{L=1}^{\infty} (2L+1)^d \, (Ke^{M-\varepsilon\beta/2})^{[\log L]+1}$$

then we see by the dominated convergence theorem that $z(\varepsilon, \beta)$ converges to zero as $\beta \to \infty$. Thus the proposition follows. Q.E.D.

Remark. The constant β_0 appearing in the proof of Proposition 2 increases with the dimension of P. It only gives an upper bound for the critical inverse temperature. This means that our approach requires stronger conditions on the inverse temperature in higher dimensions.

Proposition 3 is a consequence of the following series of geometrical statements, see Lemma 4, Corollary 1 and Corollary 2. The proof of Lemma 4 below is actually rather long and tedious, however intuitive meaning might be clear and therefore we shall omit it.

For subsets A and B of $\mathbb{Z}^d$, we introduce the sets $\partial_A B$ and $\partial_A^* B$ by

$$\partial_A B \triangleq \{ i \in A \mid \exists j \in B \text{ s.t. } \|i-j\|=1 \}$$

$$\partial_A^* B \triangleq \{ i \in A \mid \exists j \in B \text{ s.t. } |i-j|=1 \}$$

Lemma 4. Let V be any subset of $\mathbb{Z}^d$. Take any connected component D of V and $*$connected component E of $\mathbb{Z}^d \setminus V$.

(i) The sets $\partial_E D$ and $\partial^*_D E$ are, if non-empty, $*$connected and connected, respectively.

(ii) Any connected (respectively $*$connected) path from any $x \in D$ to any $y \in E$ intersects the set $\partial_E D$ (respectively $\partial^*_D E$).

For a finite subset B of $\mathbb{Z}^d$ $(d \geqslant 2)$ we set A (respectively A') the unique infinite $*$connected (respectively connected) component of $\mathbb{Z}^d \setminus B$. We introduce the notations $\partial_{\mathrm{Ext}} B$ and $\partial^*_{\mathrm{Ext}} B$ by

$$\partial_{\mathrm{Ext}} B \triangleq \partial_A B \qquad \text{and} \qquad \partial^*_{\mathrm{Ext}} B \triangleq \partial^*_{A'} B$$

If any site of B belongs to some finite $*$connected (respectively connected) component of $\mathbb{Z}^d \setminus A$, then we say that A encloses (respectively $*$encloses) B.

The following corollary is an immediate consequence of Lemma 4.

Corollary 1. Suppose $d \geqslant 2$.

(i) Let D be any finite connected subset of $\mathbb{Z}^d$. Then the set $\partial_{\mathrm{Ext}} D$ is $*$connected and $*$encloses D.

(ii) Let D be any finite $*$connected subset of $\mathbb{Z}^d$. Then the set $\partial^*_{\mathrm{Ext}} D$ is connected and encloses D.

Corollary 2. Suppose $d \geqslant 2$. Let V be any subset of $\mathbb{Z}^d$ which includes at least one infinite connected component D. Suppose all $*$connected components of $\mathbb{Z}^d \setminus V$ are finite. Then all $*$connected components of $\mathbb{Z}^d \setminus D$ are also finite. Hence D is necessarily the unique infinite connected component of V.

Proof of Corollary 2. Pick an arbitrary site not belonging to the set D. By translation, we can assume that the site is the origin O without loss of generality. Let E_L denote the union of Λ_L and all $*$connected components of $\mathbb{Z}^d \setminus V$ which meet $\partial^*_{\mathrm{Ext}} \Lambda_L$; recall $\Lambda_L = [-L, L]^d \cap \mathbb{Z}^d$. Now we take L so large that D meets $\partial^*_{\mathrm{Ext}} \Lambda_L$. Then, by Corollary 1(ii) and the definition of E_L, the set $\partial^*_{\mathrm{Ext}} E_L$ is contained in D and encloses the origin. Therefore the origin belongs to some finite $*$connected component of $\mathbb{Z}^d \setminus \partial^*_{\mathrm{Ext}} E_L$. And further, since $\partial^*_{\mathrm{Ext}} E_L \subset D$, it belongs to some finite $*$connected component of $\mathbb{Z}^d \setminus D$. Q.E.D.

Proof of Proposition 3. To simplify description, we assume again that P is the whole lattice $\mathbb{Z}^d$ $(d \geqslant 2)$. In all other cases the proof is similar. We use simple notations $V(G, \omega)$, $\xi(G, \omega)$ and $\xi^0(G, \omega)$ instead of $V_{\mathbb{Z}^d}(G, \omega)$, $\xi_{\mathbb{Z}^d}(G, \omega)$ and $\xi^0_{\mathbb{Z}^d}(G, \omega)$. Further we use the notations Λ_L and A_L defined in the proof of Proposition 2.

We pick an arbitrary element $\omega \in \bigcap_{L=1}^{\infty} A_L$. We shall show that

$$\xi^0(G_\varepsilon(\Phi), \omega) \neq \varnothing \tag{14}$$

We let Γ be any $*$connected component of $\mathbb{Z}^d \backslash V(G_\varepsilon(\Phi), \omega)$. We can take a sufficiently large L with $\Lambda_L \cap \Gamma \neq \varnothing$. Then the definition of A_L implies that $\sharp\Gamma$ is at most $\log L$. Thus

$$\sharp\Gamma < \infty \qquad \text{for all} \quad *\text{connected components } \Gamma \text{ of } \mathbb{Z}^d \backslash V(G_\varepsilon(\Phi), \omega) \tag{15}$$

Especially if we note that $\omega \in A_1$, we see that $\mathbb{Z}^d \backslash V(G_\varepsilon(\Phi), \omega)$ has no $*$connected component Γ such that $\Gamma \cap [-1, 1]^d \neq \varnothing$. Thus we find that $O \in V(G_\varepsilon(\Phi), \omega)$. In particular, we can take connected component D of $V(G_\varepsilon(\Phi), \omega)$ which contains the origin.

Now we claim that

$$\sharp D = \infty \tag{16}$$

In order to show the claim (16), we suppose that $\sharp D < \infty$. Then applying Corollary 1(i), we find that $\partial_{\mathrm{Ext}} D$ is $*$connected and $*$encloses the origin. Since $D \subset V(G_\varepsilon(\Phi), \omega)$, we observe that $\partial_{\mathrm{Ext}} D \subset \mathbb{Z}^d \backslash V(G_\varepsilon(\Phi), \omega)$.

On the other hand, the definition of A_L implies that $\sharp\Gamma \leqslant \log |j|$ for each $*$connected subset Γ of $\mathbb{Z}^d \backslash V(G_\varepsilon(\Phi), \omega)$ and each site $j \in \Gamma$. Therefore there exists no $*$connected subsets of $\mathbb{Z}^d \backslash V(G_\varepsilon(\Phi), \omega)$ which $*$encloses the origin, in contradiction to the above. Hence we get (16).

By virtue of the properties (15) and (16), we can apply Corollary 2 with $V = V(G_\varepsilon(\Phi), \omega)$. Consequently we conclude that D is just the unique infinite connected component $\xi(G_\varepsilon(\Phi), \omega)$ of $V(G_\varepsilon(\Phi), \omega)$ such that all $*$connected components of $\mathbb{Z}^d \backslash \xi(G_\varepsilon(\Phi), \omega)$ are finite sets. That is, $\xi^0(G_\varepsilon(\Phi), \omega) = D \neq \varnothing$. Q.E.D.

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